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Sequence Spaces and Analytic Sets

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Let X be a separable Banach space and S a closed subspace of $c_0(X)$ or $l^1(X)$. In S we single out the sequences $s \neq 0$ all of whose coordinates belong to some (complex) line L, and call the lines L *asymptotes* of S. It is clear that the union of all the asymptotes is an analytic (or Suslin) set in X, and our main result is the converse. © 1991 Academic Press, Inc.

INTRODUCTION

THEOREM. *Let $\mathscr{A}$ be an analytic set in the unit sphere of X. Then there are a closed linear subspace S_0 of $c_0(X)$, and a closed linear subspace S^1 of $l^1(X)$, whose asymptotes are just the lines meeting $\mathscr{A}$.*

We write T_0 and T^1 for the two statements, and present the proof of T^1 first, as it is quite brief.

Proof of T^1. Let $\mathscr{N}$ be the topological space (Baire null-space) of sequences $m = (m_1, m_2, ..., m_k, ...)$ of positive integers. The distance between different sequences m and m' is 2^{-r}, r being the first place at which $m_k \neq m'_k$. When $(m_1, ..., m_k)$ is a finite sequence, then $(m_1, ..., m_k, \cdot)$ is its completion by 1's: $(m_1, ..., m_k, 1, 1, 1, ...)$. We form a Banach space isometric with l^1, with basic elements $[m_1], [m_1, m_2], ..., [m_1, m_2, ..., m_k], ...$; the norm of $[m_1, m_2, ..., m_k]$ is 2^{-k} $(k = 1, 2, 3, ...)$.

Let F be a bounded, Lipschitz mapping of $\mathscr{N}$ into X. We define an operator, written $F*$, from l^1 into $l^1(X)$ as

$$F*[m_1] = F(m_1, \cdot)[m_1],$$

$$F*[m_1, m_2] = F(m_1, m_2, \cdot)[m_1, m_2] + (F(m_1, m_2, \cdot) - F(m_1, \cdot)) \cdot [m_1],$$

$$F*[m_1, m_2, m_3] = F(m_1, m_2, m_3, \cdot)[m_1, m_2, m_3] + (F(m_1, m_2, m_3, \cdot) - F(m_1, m_2, \cdot))([m_1] + [m_1, m_2]), \quad \text{etc.}$$

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If $\|F\|_\infty \leqslant c$ and $\|F(m)-F(m')\| \leqslant cd(m, m')$ then $F*$ has norm at most $3c$. To each element $m=(m_1, ..., m_k, ...)$ of $\mathcal{N}$ we attach an element of norm 1

$$\delta(m)=[m_1]+[m_1, m_2]+\cdots+[m_1, m_2, ..., m_k]+\cdots$$

and observe the identity $F*\delta(m)=F(m)\cdot\delta(m)$, an element of $l^1(X)$. The curious rule for multiplication was introduced in [4]. Let Y be the subspace of $l^1(X)$ spanned by elements of the form $x\,\delta(m)$, $x\in X$, $m\in\mathcal{N}$. The multiplication defined above, using a *scalar-valued* function u maps Y into itself, since $u*(x\cdot\delta(m))=u(m)\cdot x\,\delta(m)$. The multiplication is *strongly continuous* in the following sense.

Whenever $u_n\in \operatorname{Lip}(X)$, $\|u_n\|\leqslant C$ in $\operatorname{Lip}(X)$, and $u_n\to 0$ pointwise on $\mathcal{N}$; then $u_n*y\to 0$ for any y in Y. Consequently [2]

LEMMA. *The equation $u*y=0$ implies that $y=0$, or u has a zero in $\mathcal{N}$.*

We improve this as follows. Let $y\neq 0$ in Y, and let $I(y)$ be the ideal of functions in $\operatorname{Lip}(X)$ such that $u*y=0$. Then $I(y)$ has a zero m_0 in $\mathcal{N}$, i.e., $u(m_0)=0$ for every u in $I(y)$. Otherwise, in view of the separability of $\mathcal{N}$, there would be a sequence (u_j) in $I(y)$, with no common zero. Then $w=\sum a_j\,|u_j|^2$ belongs to $I(y)$ for certain scalars $a_j>0$, but $w>0$ everywhere, contradicting the lemma.

Let now F be X-valued and $\|F(m)\|=1$ everywhere. We let S^1 be the closed linear span of $\{F(m)\,\delta(m),\ m\in\mathcal{N}\}$ in $l^1(X)$. Clearly each value $F(m)$ is an asymptote, since $F(m)\cdot\delta(m)=F*\delta(m)$ has all of its coordinates in the line through $F(m)$. We proceed to the reverse inclusion, beginning with some element $y=\lim F*\alpha_n$, where the α_n's belong to the scalar version of Y; and $y\neq 0$ has 1-dimensional range L. (If the sequence (α_n) were convergent, matters would be simpler.)

Let m_0 be a zero for the ideal $I(y)$, and let W be a neighborhood of m_0 on which $\|F-F(m_0)\|<\frac{1}{4}$; let u be an element of $\operatorname{Lip}(X)$, such that $u(m_0)=1$ and $u=0$ outside W.

We claim that $u*\alpha_n$ converges in l_1. To see this we choose a linear functional x^*, such that $x^*(F(m_0))=1$, and $\|x^*\|=1$. Then $|x^*\circ F-1|<\frac{1}{2}$ on W. The sequence $(x^*\circ F)*\alpha_n$ converges, hence the sequence $(x^*\circ F)\cdot u*\alpha_n$ also converges. There is an element v of $\operatorname{Lip}(X)$ such that $v\cdot(x^*\circ F)\cdot u=u$, whence $u*\alpha_n$ has a limit β. Moreover $F*\beta=\lim(Fu)*\alpha_n=u*y\neq 0$ so $\beta\neq 0$; and since $F*\beta=u*y$ we see that the coordinates of $F*\beta$ are contained in the line L containing the coordinates of y. Whenever $x^*\in L^\perp$ we have $(x^*\circ F)*\beta=0$, whence the functions x^*F, $x^*\in L^\perp$ have a common zero in $\mathcal{N}$, i.e., L meets $F(\mathcal{N})$. (The same reasoning confirms that L meets $F(m_0)$.)

It is well known that $\mathcal{N}$ admits a Lipschitz mapping F onto any Polish space of finite diameter, and therefore (by a projection) onto a Suslin set

$\mathscr{A}$ in the unit sphere of X. Thus there is a closed subspace of $l^1(X)$, whose asymptotes are just the lines meeting $\mathscr{A}$.

Building Subspaces of $c_0(X)$

The basic idea from topology goes back to Mazurkiewicz and Sierpiński [6]; in analysis the idea for building subspaces derives from a theorem of Wiener (1924) [5, p. 42; 3], but is well disguised in the present work. The work of this section might appear to aim for greater generality than is necessary for the main theorem; this point is discussed in the last section. Let M be a compact metric space. We find a subspace $E^\infty(X)$ of $l^\infty(X)$, depending on M as well, with the following properties:

(i) $\mathrm{Lip}(M)$ operates in E^∞.

(ii) If $\alpha \neq 0$ belongs to E^∞, then the ideal $I(\alpha)$ in $\mathrm{Lip}(M)$ has an uncountable zero-set. (We recall that $u \in I(\alpha)$ means $u * \alpha = 0$.)

After $E^\infty(X)$ is constructed, we explain how to attain the same objectives with a subspace $E_0(X)$ of $c_0(X)$. (Both steps could be combined, at the cost of additional obscurity.) First we need two sequences, $(h_p)_1^\infty$ and $(u_j)_1^\infty$. For the first we require that

(a) $h_1 = 1$, $0 \leqslant h_p \leqslant 1$, and $h_p \in \mathrm{Lip}(M)$.

(b) The support of h_p has diameter $o(1)$ as $p \to +\infty$.

(c) $\limsup h_p = 1$ throughout M.

For the second we require that

(d) $|u_j| \leqslant 1$, $|u_j(m) - u_j(m')| \leqslant d(m, m')$.

(e) (u_j) is uniformly dense in the set of continuous functions defined by $|u| \leqslant 1$, $|u(m) - u(m')| \leqslant d(m, m')$.

Let E be the linear space spanned by X-valued measures in M of the form $\psi \cdot \lambda$, where λ is a continuous (diffuse) scalar measure in M, and ψ is a bounded Borel function on M to X. Each measure μ has a sequence of "moments"

$$c_{pj}(\mu) = \int h_p u_j \, d\mu$$

and we define $\|\mu\|' = \sup \|c_{pj}(\mu)\|$. Since $h_1 = 1$, we see by (d) and (e) that $\|\mu\|' > 0$ unless $\mu = 0$. We define $E^\infty(X)$ to be the completion of E for the norm $\|\mu\|'$. Since $\|u_j \mu\|' \leqslant \|\mu\|'$ for each $\mu \in E$ and each u_j, we get a representation of $\mathrm{Lip}(M)$ in E, which we denote by $u * \mu$. The main idea in the crucial property (ii) of E^∞ is the next

LEMMA. *Let $\mu \neq 0$ in E^{∞} and $m_0 \in M$. Then there is some u in* Lip(M), *such that $u(m_0)=0$ and $u * \mu \neq 0$.*

Proof. In the opposite case, the formula $u * \mu \equiv u(m_0) \cdot \mu$ would be true. Now $\mu = \lim \mu_k$, where (μ_k) is a sequence of vector measures in E. For each fixed k and j, we have $\lim_p c_{pj}(\mu_k)=0$, so the same must be true with μ in place of μ_k. Now each h_p is in Lip(M), and $h_1=1$. Hence $c_{pj}(\mu)= c_{1j}(h_p * \mu)=h_p(m_0)\, c_{1j}(\mu)$. But $\limsup h_p(m_0)=1$, while $\lim_p c_{pj}(\mu)=0$. Hence $c_{1j}(\mu)=0$, and again $c_{pj}(\mu)=h_p(m_0)\, c_{1j}(\mu)=0$. Thus $\mu=0$ as claimed.

To proceed we need some properties of ideals in Lip(M). When S is a subset of M, then $u \in I(S)$ means $u=0$ on S; $u \in J(S)$ means $u=0$ on a neighborhood of S; and $I_0(S)$ is the norm-closure of $J(S)$. First of all, $u \in I(S)$ implies that $u^2 \in I_0(S)$. In fact $u^2 = \lim w_N$, where

$$w_n(m)=0, \qquad |u(m)| \leqslant N^{-1}$$

$$w_n(m)=u^2(m)-N^{-1}u^2(m)\,|u(m)|^{-1}, \qquad |u(m)| \geqslant N^{-1}.$$

From the identity $4uv=(u+v)^2-(u-v)^2$ we obtain (symbolically) $I(S)\cdot I(S) \subseteq I_0(S)$.

For any ideal I, not necessarily closed, we denote by $\zeta(I)$ its set of common zeroes in M, and assert that $I \supseteq J(\zeta(I))$. In fact, let $u \in J(\zeta(I))$, so that $u=0$ on an open set V containing $\zeta(I)$. Then I contains an element v, positive on $M \backslash V$ (and therefore positive on ∂V). The equation $u=gv$ admits a solution g in Lip(M)—determined by the condition that $g=0$ on V, and $g=uv^{-1}$ on $M \backslash V$.

To prove property (ii) of the space $E^{\infty}(X)$, let $u, v \in I(\zeta)$, where ζ is the zero-set of $I(\alpha)$. Then $uv \in I_0(\zeta)$, and therefore $uv \in I(\alpha)$, because $I(\alpha)$ is closed and $I(\alpha) \supseteq J(\zeta)$. Let z_1 be an isolated point of ζ and ζ' the remainder of ζ; let u_1, v_1 belong to $I(z_1)$, and $w \in J(\zeta')$. Then $u_1 v_1 w \in I_0(\zeta')\, I_0(z_1) \subseteq I_0(\zeta)$, whence $u_1 v_1 w * \alpha = 0$. By the lemma, $v_1 w * \alpha = 0$, $w * \alpha = 0$. Thus $\zeta = \zeta'$, a contradiction, proving that ζ must be perfect.

We can now explain how to replace E^{∞} by a subspace E_0 of $c_o(X)$. Let K be the subset of $C(M)$ defined by

$$|u| \leqslant \tfrac{1}{2}, \qquad |u(m)-u(m')| \leqslant d(m, m').$$

Then there is a null sequence $\sigma=(v_k)$ in $C(M)$, such that $|v_k| \leqslant \frac{2}{3}$ and $K \subseteq \overline{c_0}(\sigma)$. Here is a sketch of the demonstration. First there is a finite set $F_1 \subseteq K$, and a compact set $K_1 \subseteq B(2^{-1}\cdot 10^{-1})$ such that $K \subseteq F_1 + K_1$. Then K_1 can be approximated in the same way, etc. Eventually we obtain finite sets $F_j \subseteq B(2^{-1}\cdot 10^{1-j})$ such that $K \subseteq {}_1 + F_2 + \cdots + F_j + \cdots$. The null-sequence is then $\frac{4}{3}F_1 \cup 2^{j+1}F_j$ and it works because $\frac{3}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$.

Let $(\psi_\nu)_1^\infty$ be an enumeration of all the elements in the multiplicative semi-group spanned by σ, i.e., all products out of σ. Since σ is a null-sequence in $B(0, \frac{2}{3})$, we have $\|\psi_\nu\| \to 0$. We define now

$$\|\mu\|'' = \sup \left\| \int h_p \psi_\nu \, d\mu \right\|, \qquad \mu \in E.$$

Taking care to write each product only once we see that the completion E_0 is a closed subspace of $c_o(X)$. Now $\|\psi_\nu \mu\|'' \leqslant \|\mu\|''$, whence $\|\psi\mu\|'' \leqslant \|\mu\|$ for each ψ in K. This explains the operation of Lip(M) in E_0, and (ii) is obtained as before.

A Construction by Mazurkiewicz and Sierpiński [6]

Let $\mathscr{S}$ be a Suslin set in a compact metric space M_1. Then there is a compact metric space M, and a continuous map Φ of M into M_1 such that

(i) $\Phi^{-1}(y)$ is uncountable for each y in $\mathscr{S}$,

(ii) $\Phi^{-1}(y)$ is denumerable for each y in $M_1 \backslash \mathscr{S}$.

Before writing out the details, we remark that this does not provide exactly what is needed later; a certain adjustment is made afterwards. We write $\mathscr{I}$ for the irrationals in (0, 1), so there is a continuous map ϕ of $\mathscr{I}$ onto $\mathscr{S}$. Since $\mathscr{I}$ is homeomorphic with $\mathscr{I} \times \mathscr{I}$, this can be accomplished so that $\phi^{-1}(y)$ is uncountable for each y in $\mathscr{S}$. Let M be the closure of the graph of ϕ, i.e., the closure of the set $\{(t, \phi(t)) : t \in \mathscr{I}\}$ in $R \times M_1$, and Φ the projection of M into M_1. If $(t, y) \in M$ and $y \notin \mathscr{S}$ then t must be rational; this yields (ii) at once. The space M is provided with a metric, so that Φ is a contraction map.

We want a variant of this for Suslin sets $\mathscr{S}$ in $X \backslash \{0\}$, with compact closure, namely a mapping Φ such that

(i′) $\Phi^{-1}(L)$ is uncountable for each line L meeting $\mathscr{S}$,

(ii″) $\Phi^{-1}(L)$ is denumerable for all other lines.

We can write $\mathscr{S} = \bigcup \mathscr{S}_i$, where each $\mathscr{S}_i$ is contained in some ball $B(x_i, \frac{1}{4}\|x_i\|)$, $x_i \in \mathscr{S}$. Let x_i^* be a linear functional of norm 1, such that $x_i^*(x_i) = \|x_i\| > 0$. We map $\mathscr{S}_i$ to a Suslin set $\mathscr{S}_i'$ by a mapping $h_i(x) = 2^{-i} x_i^*(x)^{-1}\, x$, so that all the points in $\mathscr{S}_i$ are in the annular region $\frac{4}{5} 2^{-i} < \|x\| < \frac{4}{3} 2^{-i}$. The set $\mathscr{S}' = \bigcup \mathscr{S}_i'$ again has compact closure, and on each $\mathscr{S}_i'$, $x_i^*(x) \equiv 2^{-i}$. Clearly $\mathscr{S}'$ meets exactly the same lines L as $\mathscr{S}$, and each line L meets the *closure* of $\mathscr{S}'$ only countably often, since $\mathrm{cl}(\mathscr{S}') = \bigcup_1^\infty \mathrm{cl}(\mathscr{S}_i') \cup \{0\}$. The mapping Φ, constructed as before, certainly has property (i′). As for (ii′), we know that $L \cap \Phi(M)$ is equal to $L \cap \mathrm{cl}(\mathscr{S}')$, and the latter set is at most denumerable. But $\Phi^{-1}(y)$ is denumerable unless $y \in \mathscr{S}'$, and (ii′) follows from these two facts.

Conclusion

We can now prove T_0 for the special case of sets $\mathscr{S}$ with compact closure (not necessarily in the unit sphere.) Let M and Φ be the metric space and continuous mapping just constructed and $E_0(X)$ the subspace of $c_o(X)$ over M. In $E_0(X)$ we define S as the closed, linear span of vector measures $\Phi \cdot \lambda$, with λ a scalar continuous measure in M.

For each y in $\mathscr{S}$, the line L through y has an uncountable pre-image $\Phi^{-1}(y)$. That set carries a continuous probability measure λ, and then $\Phi \cdot \lambda$ belongs to S and has all coordinates in L. ($\Phi \cdot \lambda \neq 0$, since $\Phi^{-1}(0)$ is denumerable.)

Conversely, let $y \neq 0$ be an element of S, all of whose coordinates belong to a complex line L. Then $y = \lim \Phi \cdot \lambda_n$, where λ_n are scalar-valued continuous measures in M. We first produce a scalar Lipschitz function u, such that $\lim u \cdot \lambda_n = \beta \neq 0$, using the metric for scalar measures analogous to that introduced for vector measures. Since $y \neq 0$, there is a linear functional x_0^* such that $x_0^*(y)$ has a coordinate $\neq 0$. Thus we can choose $u = x_0^*(\Phi)$. Since Φ is a contraction map of M into X, we can see that $\Phi * \beta$ belongs to $E_0(X)$, and that $\Phi * \beta = \lim u\Phi * \lambda_n$, whence $\Phi * \beta = u * y$. Moreover u^*y has coefficients in L, since these can be calculated by a limit process from the coefficients of y, by means of the sequence σ (see the definition of the norm $\|\mu\|''$). For any linear functional x^* the scalar sequences $x^*\Phi * \beta$ and $x^*(u * y)$ coincide; hence $x^*\Phi * \beta = 0$ whenever $x^* \in L^\perp$. Thus $x^*\Phi$ belongs to the ideal $I(\beta)$ of $\mathrm{Lip}(M)$, and that ideal has an uncountable zero-set. It follows that $\Phi^{-1}(L)$ is uncountable, whence L meets $\mathscr{S}$, as required.

For the general case in which $\mathscr{A}$ is a Suslin set in the sphere of X, we observe that there is a compact, one-one linear transformation B of X into a separable space Y, and specify $\|B\| < 1$. The Suslin set $\mathscr{S} = B(\mathscr{A})$ therefore has compact closure in Y and avoids 0. We apply to it the operation just completed, obtaining a certain closed subspace $S \subseteq c_o(Y)$. It will be convenient to write elements of $c_o(X)$ or $c_o(Y)$ as sequences (x_N) or (y_N). The elements of S_0 are now defined by the condition $(By_N) \in S$.

Suppose $s = (y_N) \neq 0$ and each $y_N \in L$. Then the sequence $(By_N) \neq 0$ and each $By_N \in B(L)$. Hence the line $B(L)$ meets $\mathscr{S}$, and L must meet $\mathscr{A}$, since B is one-one on X. Conversely, let $a \in \mathscr{A}$, whence $Ba \in \mathscr{S}$ and there is a non-zero scalar sequence (c_N) such that $(c_N Ba) \in S$. Thus $(c_N a) \in S_0$, and the asymptotes of S_0 are the lines meeting $\mathscr{A}$.

Remarks. As T_0 has been obtained with so many detours, it is worthwhile to explain some of them. To construct the spaces E^{∞} and E_0 we need a sequence of sets V_j such that $\mathrm{diam}(V_j) \to 0$, $M = \limsup V_j$. This is not possible for the unit ball or the unit sphere of an infinite-dimensional Banach space [1]. The space $\mathrm{Lip}(M)$ is separable in the supremum norm only when the metric space is totally bounded; the separability is indispensable for E_0. In the model construction of [6], the metric space M is a real interval. This is possible because M_1 has finite topological dimension. It appears that $M = [0, 1]$ would be possible if and only if $\mathrm{cl}(\mathscr{S}) \setminus \mathscr{S}$ is a countable union of sets of dimension 0 [7]; it does not seem possible to omit the special case in which $\mathscr{A}$ has compact closure.

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